LETTER TO THE EDITOR

The Kasteleyn model and a cellular automaton approach to traffic flow

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Abstract. We propose a bridge between the theory of exactly solvable models and the investigation of traffic flow. By choosing the activities in an appropriate way, the dimer configurations of the Kasteleyn model on a hexagonal lattice can be interpreted as spacetime trajectories of cars. This then allows for a calculation of the flow-density relationship (fundamental diagram). We further introduce a closely related cellular automaton model. This model can be viewed as a variant of the Nagel–Schreckenberg model in which the cars do not have a velocity memory. It is also exactly solvable and the fundamental diagram is calculated.

1. Introduction

In the past few years the investigation of traffic flow using cellular automata has become quite popular (see e.g. [1] and references therein). In contrast to the hydrodynamical description [2–4] this approach is microscopic in the sense that individual cars can be distinguished in a similar way to the follow-the-leader models [5–7]. The simplicity of this approach allows for a number of new applications which cannot be incorporated easily in the more complex theories of traffic dynamics. Particularly, large-scale computer simulations are now possible [8, 9] which are effective even in the case of complicated street networks.

   In this paper we propose a different approach which uses methods from statistical mechanics. It is well known that a number of exactly solvable models of statistical mechanics have a graphical interpretation in terms of (closed) graphs. We suggest the interpretation of such graphs as trajectories of cars in traffic. Using the known partition function of an exactly solvable free-fermion model, the Kasteleyn dimer model on a hexagonal lattice, we are then able to calculate the properties of the corresponding traffic model. In this way we not only obtain information about the fundamental diagram (density versus flow) but also about the correlation functions. Finally, we discuss the relation of this model to the approach using cellular automata.

2. Kasteleyn model

Here we suggest and study a simple statistical two-dimensional lattice model, the configurations of which can be mapped onto the trajectories of a discrete (stochastic) one-dimensional traffic problem on a ring ('Indianapolis situation'); see figure 1. Our traffic

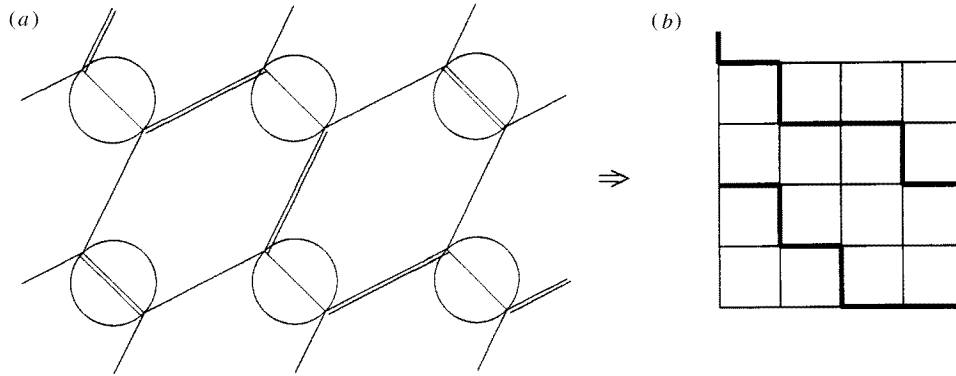‖ Permanent address: Institute of Mechanics, Bulgarian Academy of Sciences, 1113 Sofia, Bulgaria.

Figure 1. Representation of the hexagonal lattice as a decorated quadratic lattice (*a*), and its mapping onto the square lattice (*b*). The encircled pairs of sites of the decorated lattice map on a single site of the square lattice, the vertical (horizontal) dimers on the former lattice map onto space (time) steps of the trajectories on the latter.

model has the following properties: (i) the cars do not collide, the minimum distance between any two cars being not less than one lattice spacing; (ii) the velocities of the cars are (in principle) unlimited. The principal difference between the cellular automaton models and the present approach is that the statistical weights are ascribed globally to each allowed set of trajectories, rather than determined locally by the transition probabilities from one space configuration to another.

The statistical model is actually an interpretation of the dimer model on the hexagonal lattice with anisotropic activities, which has been suggested by Kasteleyn [10] and studied in a series of papers; see [11] and references therein. The hexagonal lattice we represent as a decorated square lattice with two sites in each vertex shown by a circle in figure 1(*a*). We consider the horizontal direction of the square lattice as the space axis and the vertical one downwards as the time axis of the traffic problem as usual in traffic theory. The above interpretation allows us to define a one-to-one mapping of each dimer configuration on the decorated lattice, figure 1(*a*), onto a set of trajectories on the square lattice, figure 1(*b*). Under that mapping the circles of the decorated lattice are mapped onto the sites of the square lattice; a horizontal dimer on the decorated lattice corresponds to a unit space step of some trajectory, and a vertical dimer corresponds to a unit time step. The fact that the Kasteleyn model belongs to the class of free-fermion models ensures continuity and non-intersection of the trajectories. The number of trajectories and the average velocity of the cars are controlled by the activities x and t of the horizontal and vertical edges of the lattice, respectively; the activity of the slanted edges in the circles, figure 1(*a*), is set to unity. The statistical weight of each allowed set of trajectories is determined by the total number of horizontal and vertical unit steps in it. Thus, the generating (or partition) function of the model on a finite square lattice with L columns and M rows is given by

$$Z_{L,M}(x, t) = \sum_{\{C\}} x^{N_x(C)} t^{N_t(C)} . \tag{1}$$

Here the summation runs over the set of all allowed dimer configurations on the hexagonal lattice. For periodic boundary conditions, the solution of the Kasteleyn model in the

thermodynamic limit is [11]

$$f_\infty = \lim_{M,N \to \infty} \frac{1}{ML} \ln Z_{L,M}(x,t) = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} d\alpha \, d\beta \, \ln|1 - te^{i\alpha} - xe^{i\beta}| \,. \tag{2}$$

The phase diagram of the model has four regions in the plane $0 \leqslant t < \infty$, $0 \leqslant x < \infty$ [13], denoted here by A, B, C and D. The phase in region A,

$$(x,t) \in A = \{t > 1 + x\} \tag{3}$$

describes stopped, close-packed cars. There is a trivial empty phase in region B,

$$(x,t) \in B = \{x < 1, t < |1 - x|\} \tag{4}$$

and the phase in C

$$(x,t) \in C = \{x > 1, t < |1 - x|\} \tag{5}$$

corresponds to the zero-density limit of cars moving at infinite speed. Normal trajectories of cars at all intermediate densities exist in the region $(x,t) \in D$,

$$(x,t) \in D = \{|1 - x| < t < |1 + x|\} \tag{6}$$

which motivates our interest only in that non-trivial case.

The integration over α in equation (2) can be easily performed by using the Jensen formula [14]. In region D the result is

$$f_\infty = \Delta \ln(t) + \frac{1}{2\pi} \int_{\pi\Delta}^{\pi} d\beta \, \ln(1 - 2x \cos(\beta) + x^2) \tag{7}$$

where

$$\Delta = \frac{1}{\pi} \arccos\left(\frac{1 + x^2 - t^2}{2x}\right). \tag{8}$$

Let us now turn to the interpretation of the observables in our model. For a finite lattice under periodic boundary conditions, the total numbers of steps in time and space can be expressed as

$$N_t(C) = MN_a(C) \tag{9}$$

and

$$N_x(C) = LN_a(C)w(C) \tag{10}$$

where N_a is the number of automobiles in configuration C, and $w(C)$ is the number of windings of each trajectory in the periodic spatial direction. Therefore, the average density of cars can be defined as

$$\rho(x,t) = \frac{1}{LM} \langle N_t(C) \rangle = t \frac{\partial}{\partial t} f_\infty(x,t) = \Delta(x,t) \,. \tag{11}$$

The rigorous definition of the average velocity v of cars is given by the average value of the ratio of space steps to time steps. However, as a first approximation we can assume

$$v = \frac{\langle N_x(C) \rangle}{\langle N_t(C) \rangle} \,. \tag{12}$$

The numerator of the above ratio is obtained by differentiation of expression (7) with respect to the activity of the space steps x:

$$\frac{1}{LM} \langle N_x(C) \rangle = x \frac{\partial}{\partial x} f_\infty(x,t) = \frac{1}{2}(1 - \Delta(x,t)) + \frac{\text{sign}(x-1)}{2}(1 - K(x,t)) \,. \tag{13}$$

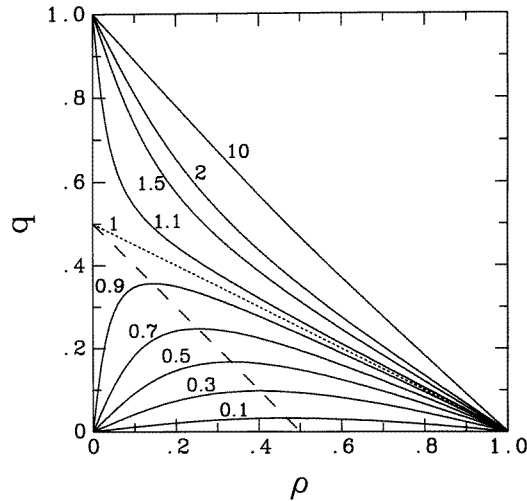**Figure 2.** The flow $q(x, \rho)$ for different values of x. Curves with $x < 1$ lie below the dotted line ($x = 1$), curves with $x > 1$ above. The broken line indicates the location of the maximum flow.

Here we have introduced the function

$$K(x, t) = \frac{2}{\pi} \arctan\left(\frac{x+1}{|x-1|} \sqrt{\frac{t^2 - (1-x)^2}{(1+x)^2 - t^2}} \right).$$

(14)

To obtain the fundamental traffic flow–density diagram, we start with the definition of the flow

$$q = v\rho$$

(15)

which, in combination with (11)–(13), gives

$$q(x, t) = \frac{\langle N_x(C) \rangle}{LM}.$$

(16)

The parameter t plays an auxiliary role in our model and we will express it in terms of the density ρ and activity x by using the relation $\Delta(x, t) = \rho$, see (8) and (11). Thus, for the traffic flow $q(x, \rho)$ we obtain the following explicit expression:

$$q(x, \rho) = \frac{1 - \rho}{2} + \frac{\text{sign}(x - 1)}{2\pi} \left[\pi - \arccos\left(\frac{(x-1)^2(1 + \cos(\pi\rho))}{x^2 - 2x\cos(\pi\rho) + 1} - 1 \right) \right].$$

(17)

Note that the variable x controls the average velocity v_m of a single car. Indeed, by considering configurations with only one trajectory, one can readily see that $v_m = x/t$.

The flow–density diagram at different fixed values of x is shown in figure 2. A remarkable feature of that diagram is the existence of two qualitatively different traffic-flow regimes. For $x < 1$ the flow reaches its maximum value

$$q(x, \rho_{\max}) = \tfrac{1}{2} - \rho_{\max}$$

(18)

at the density

$$\rho_{\max} = \frac{1}{\pi} \arccos(x).$$

(19)

In the other regime, when $x > 1$, the traffic flow decreases monotonically with the increase of ρ, from $q(x, 0) = 1$ to $q(x, 1) = 0$.

The phenomenon of self-organized criticality in the transport flow is clearly manifested in the asymptotic behaviour of the pair correlation functions both in space and time.

Obviously, the correlations between time and space steps of the trajectories in our model correspond to dimer–dimer correlations in the original model of close-packed dimers on the hexagonal lattice. The latter can be obtained and studied by using the technique of Fisher and Stephenson [12]. In the transport-flow problem one is interested mainly in the correlations between cars passing a fixed position in space at two moments in time, say, $t = 0$ and $t = T$, as well as in the correlations between cars at the same moment in time, but at a distance R apart. The former case corresponds to the correlation function for two vertical dimers in the same row of the decorated square lattice, and the latter case corresponds to two horizontal dimers in the same column, see figure 1. Due to the symmetry of the problem, here we give the expression for the temporal correlations only. By using the method of [12], for the correlation function $K_t(T)$ one obtains in the thermodynamic limit

$$K_t(T) = -\left| \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} d\alpha \, d\beta \frac{e^{i\alpha+iT\beta}}{1 - te^{i\alpha} - xe^{i\beta}} \right|^2 . \tag{20}$$

The simple analytical structure of the integrand in the complex plane permits the exact evaluation of the above integral:

$$K_t(T) = -\frac{\sin^2(\pi T \Delta')}{(\pi x T)^2} . \tag{21}$$

Here $\Delta' \equiv \Delta'(x, t) = \Delta(t, x)$.

Similarly, for the spatial correlation function we obtain the asymptotic behaviour $K_s(R) \propto R^{-2}$ at large R. The power-law decay of both temporal and spatial correlation functions reflects the phenomenon of self-organized criticality in our model. However, the following questions arise. (i) Are the obtained critical exponents universal for the class of statistical models of traffic? (ii) Will they be the same for the cellular automaton models?

3. Comparison with the cellular automaton approach

The interpretation of the above results as a cellular automaton with local rules is not straightforward. The summation over all allowed dimer configurations corresponds to a global analysis of the statistics and there are no simple probabilistic *local* rules yielding the same distribution as the partition function (1). But it is possible to define a model with very similar properties as is shown in the following.

Consider a probabilistic process according to the following rules. One time step consists of an update of the positions of all cars in a parallel way. If a car has n empty sites in front of it (next car at $n + 1$) there are at most n partial movements of one site possible for this car. The decision of how far the car finally moves depends on a series of random numbers yielding with probability p a '1' and with $q = 1 - p$ a '0'. The car moves as long as 1's are chosen. When a 0 occurs the movement stops until, in the next time step, this process starts anew. So if the first chosen random number is a 0 the car does not move at all. If the series has l succeeding 1's the car ends up at the lth site (possibly directly behind the car ahead if it does not move itself in the same time step).

An important feature of these rules is that no memory for the velocity is needed. On the other hand this can lead to unphysical features through enormous velocity fluctuations from one time step to the next. Nevertheless, one can define a mean maximum velocity of a free driving car according to the rules just mentioned which is simply given by $\bar{v} = p/q$. This is the average distance a single car moves on a free road without other cars.

In the following we show that the mean-field ansatz already gives the exact stationary state of the cellular automaton for any system of size L and number of cars N. The exactness of the mean-field solution has the consequence, in this case, that all possible states of the system are equally probable.

We characterize a state of the system at time t only by the number d_i $(i = 1, \ldots, N)$ of free sites between car i and car $i + 1$ since no other quantities are needed. For the ring geometry this means that d_N is the distance between car N and car 1. It is then rather simple to write down the master equation for the time evolution of the system

$$P_{t+1}(\{d_i\}) = \sum_{s_1=0}^{d_N} \sum_{s_2=0}^{d_1} \cdots \sum_{s_N=0}^{d_{N-1}} \prod_{i=0}^{N} [p^{s_i}(\delta_{d_i,s_{i+1}} + (1 - \delta_{d_i,s_{i+1}})q)] P_t(\{d_i - s_i + s_{i+1}\}) . \tag{22}$$

The variable s_i denotes the distance car i drives in the timestep $t \to t + 1$. The Kronecker δ takes into account the fact that if car i ends up directly behind the next car $i + 1$ (i.e. $s_{i+1} = d_i$) it does not have the 'deceleration' factor q. This is the main difference to the dimer model where one has this factor q in any case (i.e. t in the other notation).

The static solution of (22) is simply given by $P(\{d_i\}) \equiv 1/\binom{L}{N}$, i.e. asymptotically all probabilities are equal. This can be seen easily from

$$1 = \sum_{s_1=0}^{d_N} \sum_{s_2=0}^{d_1} \cdots \sum_{s_N=0}^{d_{N-1}} \prod_{i=0}^{N} [p^{s_i}(\delta_{d_i,s_{i+1}} + (1 - \delta_{d_i,s_{i+1}})q)] \tag{23}$$

which follows from the identity

$$1 = \sum_{s_i}^{d_{i-1}} p^{s_i}(\delta_{i-1,s_i} + (1 - \delta_{d_{i-1},s_i})q) \tag{24}$$

after interchanging product and sums. The probability f_n for a car to drive exactly n sites in the thermodynamic limit $N \to \infty$ is given by

$$f_n = (1 - \rho)^n p^n [(1 - \rho)q + \rho] \tag{25}$$

since a car can stop by chance or when another car blocks it. For the flow in the CA one then obtains

$$q_{\mathrm{CA}}(p, \rho) = \rho \sum_{n=1}^{\infty} n f_n = \frac{\rho(1 - \rho)p}{1 - (1 - \rho)p} = \frac{\rho(1 - \rho)\bar{v}}{1 + \rho\bar{v}} . \tag{26}$$

To compare the flow (26) of the cellular automaton with the flow (18), we have to rescale (18) with a factor 2 $(q(x, \rho) \to 2q(x, \rho))$. To illustrate this, we consider the limiting case of low car density and $x \to 1$. The dotted line in figure 2 tends to 0.5 when $\rho \to 0$. This reflects the fact that each horizontal line of the lattice can be occupied by the trajectory of the single car or be empty with equal probability. In the cellular automata approach, the motion with infinite velocity is forbidden and the case $x \to 1$ corresponds to the maximal occupation of all horizontal bonds. This leads to the shift of the dotted line in figure 2 to the diagonal of the curve.

After the rescaling the maximum of both flows lie on the curve $1 - 2\rho$. Therefore we can find the relation between the parameters p and x in the two models by identifying curves which have the same maximum. This yields the identification

$$p = \frac{1 - (2/\pi)\arccos x}{(1 - (1/\pi)\arccos x)^2} . \tag{27}$$

Figure 4 shows $q_{\mathrm{CA}}(p, \rho)$ and $q(x, \rho)$ for $p = \frac{3}{4}$ and $x = \frac{1}{2}$. Although the slopes for small and high densities are different the overall agreement of the two curves is very good.
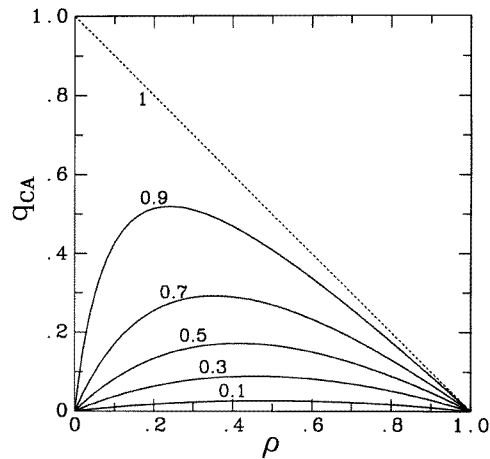
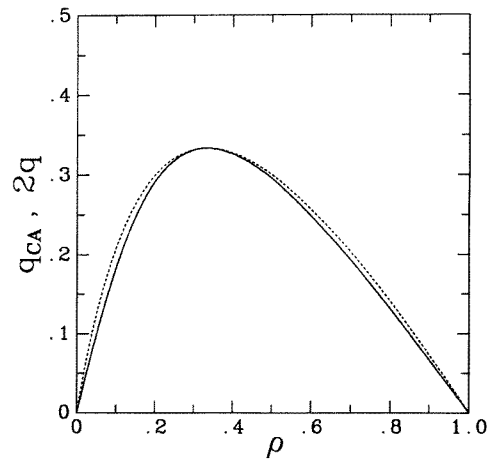**Figure 3.** The flow $q_{CA}(p, \rho)$ for different values of p.

Figure 4. Comparison of $q_{CA}(p, \rho)$ (dotted curve) and $2q(x, \rho)$ for $p = \frac{3}{4}$ and $x = \frac{1}{2}$.

In conclusion we considered the Kasteleyn model as a model of traffic flow. This model allows us to establish a bridge between the theory of exactly solvable models and the investigation of traffic flow by cellular automata. By choosing the activities in an appropriate way it is possible to interpret the dimer configurations as spacetime trajectories of vehicles. The Kasteleyn model can be related to a modified Nagel–Schreckenberg model in which the cars do not have a velocity memory. The fundamental diagram of this modified CA can be obtained exactly as well and shows good agreement with the fundamental diagram obtained for the Kasteleyn model.

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